Laplace Equation :-

Poisson's equation

$$\nabla^2 V = \frac{-\rho}{\epsilon_0} \quad \text{—— (1)}$$

$V \rightarrow$ potential

Eq. (1) in a region where $\rho = 0$, gives the Laplace equation

$$\boxed{\nabla^2 V = 0} \quad \text{—— (2)}$$

For 1.D case V ~~V(x)~~ depends only one variable x.

$$\frac{d^2 V}{dx^2} = 0$$

Soln. $\longrightarrow$ $V(x) = mx + b$.

We solve equation (2) for 3D case. ∴ We consider the rectangular coordinate system and ∇^2 for 3D is given by

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

V is function of x, y, z i.e. $V \equiv V(x,y,z)$

$$\nabla^2 V = 0$$

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \quad \text{--- (3)}$$

We use separation of variable to solve the above equation. Let us consider that $V(x,y,z)$ has solution of the form

$$V(x,y,z) = X(x)\,Y(y)\,Z(z) \quad \text{--- (4)}$$

using (4) in (3)

$$Y(y)Z(z)\frac{\partial^2 X(x)}{\partial x^2} + X(x)Z(z)\frac{\partial^2 Y(y)}{\partial y^2} + X(x)Y(y)\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

Now dividing the above equation by $X(x)Y(y)Z(z)$ we obtain

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2 Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2 Z(z)}{\partial z^2} = 0 \quad \text{--- (5)}$$

Since X, Y and Z are functions of one variable, therefore, we have used total derivative in the above equation. The above equation should hold for all possible values of independent coordinates and therefore the terms must be independent.

Thus, we write

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}=-\alpha^2,\quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2}=-\beta^2$$

$$\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=\gamma^2$$

where $\boxed{\alpha^2+\beta^2=\gamma^2}$

The above enfxessions can be written as

$$\left.\begin{aligned}\frac{d^2X(x)}{dx^2}&=-\alpha^2X(x)\\ \frac{d^2Y(y)}{dy^2}&=-\beta^2 Y(y)\\ \frac{d^2Z(z)}{dz^2}&=+\gamma^2 Z(z)\end{aligned}\right\}\text{ — (6)}$$

Solutions of above equation can be written as

$$\left.\begin{aligned}X(x)&=Ae^{i\alpha x}+Be^{-i\alpha x}\\ Y(y)&=Ce^{i\beta y}+De^{-i\beta y}\\ Z(z)&=Fe^{\gamma z}+Ge^{-\gamma z}\end{aligned}\right\}\text{ — (7)}$$

Now for $\alpha=0,\beta=0,\gamma=0$, the above enfxessions respectively can be written as

$$\left.\begin{aligned}X(x)&=A+Bx\\ Y(y)&=C+Dy\\ Z(z)&=F+Gz\end{aligned}\right\}\text{ — (8)}$$

Now for the nonzero constants, using equation (4) the solution is given by

$$V(x,y,z) = X(x)\,Y(y)\,Z(z)$$

or $$V(x,y,z) = \left(A_{\alpha\beta} e^{i\alpha x} + B_{\alpha\beta} e^{-i\alpha x}\right)\left(C_{\alpha\beta} e^{i\beta y} + D_{\alpha\beta} e^{-i\beta y}\right)\left(F_{\alpha\beta} e^{\gamma z} + G_{\alpha\beta} e^{-\gamma z}\right) \quad \text{—— (9)}$$

Eq. (9) true for $\alpha \neq 0$ & $\beta \neq 0$.

Now consider the particular cases of constants and write solutions from (9).

Case-I. ; $\alpha = 0$. and $\beta \neq 0$ $\{\Rightarrow \beta = \gamma$ since, $\gamma^2 = \alpha^2 + \beta^2\}$

$$V(x,y,z) = \left(A_{0\beta} + B_{0\beta}\, x\right)\left(C_{0\beta} e^{i\beta y} + D_{0\beta} e^{-i\beta y}\right)\left(F_{0\beta} e^{\beta z} + G_{0\beta} e^{-\beta z}\right) \quad \text{—— (10)}$$

Case-2 , $\alpha \neq 0$, $\beta = 0$

$$V(x,y,z) = \left(A_{\alpha 0} e^{i\alpha x} + B_{\alpha 0} e^{-i\alpha x}\right)\left(C_{\alpha 0} + D_{\alpha 0}\, y\right)\left(F_{\alpha 0} e^{\alpha z} + G_{\alpha 0} e^{-\alpha z}\right) \quad \text{—— (11)}$$

Case-3 , $\alpha = 0$, $\beta = 0$ $\Rightarrow \gamma = 0$, since $\gamma^2 = \alpha^2 + \beta^2$

$$V(x,y,z) = \left(A_{00} + B_{00}\, x\right)\left(C_{00} + D_{00}\, y\right)\left(F_{00} + G_{00}\, z\right) \quad \text{—— (12)}$$

Now the general solution will be given by the sum of all possible particular solutions

$$\boxed{\begin{aligned} V(x,y,z) = & \sum_{\alpha\neq 0}\sum_{\beta\neq 0} \left(A_{\alpha\beta} e^{i\alpha x} + B_{\alpha\beta} e^{-i\alpha x}\right)\left(C_{\alpha\beta} e^{i\beta y} + D_{\alpha\beta} e^{-i\beta y}\right)\left(F_{\alpha\beta} e^{\gamma z} + G_{\alpha\beta} e^{-\gamma z}\right) \\ & + \sum_{\beta\neq 0} \left(A_{0\beta} + B_{0\beta}\, x\right)\left(C_{0\beta} e^{i\beta y} + D_{0\beta} e^{-i\beta y}\right)\left(F_{0\beta} e^{\beta z} + G_{0\beta} e^{-\beta z}\right) \\ & + \sum_{\alpha\neq 0} \left(A_{\alpha 0} e^{i\alpha x} + B_{\alpha 0} e^{-i\alpha x}\right)\left(C_{\alpha 0} + D_{\alpha 0}\, y\right)\left(F_{\alpha 0} e^{\alpha z} + G_{\alpha 0} e^{-\alpha z}\right) \\ & + \left(A_{00} + B_{00}\, x\right)\left(C_{00} + D_{00}\, y\right)\left(F_{00} + G_{00}\, z\right) \end{aligned}}$$

We will discuss the specific cases of the above solution whenever solve a particular problem